Published by Institute of Physics Publishing for SISSA/ISAS

Received: *August 10, 2004*
Revised: *September 15, 2004*
Accepted: *September 15, 2004*

Effective superpotential for $\mathrm{U}(N)$ with antisymmetric matter

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Abstract: We consider an $\mathcal{N} = 1$ $\mathrm{U}(N)$ gauge theory with matter in the antisymmetric representation and its conjugate, with a tree level superpotential containing at least quartic interactions for these fields. We obtain the effective glueball superpotential in the classically unbroken case, and show that it has a non-trivial N-dependence which does not factorize. We also recover additional contributions starting at order S^N from the dynamics of $\mathrm{Sp}(0)$ factors. This can also be understood by a precise map of this theory to an $\mathrm{Sp}(2N-2)$ gauge theory with antisymmetric matter.

Keywords: Nonperturbative Effects, Anomalies in Field and String Theories, Supersymmetric Effective Theories.

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1. Introduction

Supersymmetry makes it possible to understand at least some aspects of the exact effective dynamics of confining gauge theories. In particular, for $\mathcal{N} = 1$ supersymmetric gauge theories, it is possible to determine exactly the low-energy effective F-terms and thus analyze the quantum vacuum structure and the values of the various condensates typically associated to the spontaneous breaking of some global symmetries.

Recently, a systematic approach has been devised [1]–[3] to compute such low energy effective superpotentials as functions of the glueball superfields, which are assumed to be the correct low energy fields below the confining scale of the non-abelian gauge groups. By virtue of the linearity principle and the holomorphy of such Wilsonian effective superpotentials, the information obtained in this way is exactly the same as the one obtained by integrating out the (generically massive) glueball fields S_i. See [4] for a review of this approach and a list of references.

In this paper we will consider a supersymmetric $\mathcal{N} = 1$ $\mathrm{U}(N)$ gauge theory with matter consisting only of a chiral supermultiplet $\chi^{ij} = \chi^{[ij]}$ in the antisymmetric representation, and its charge conjugate $\tilde{\chi}_{ij} = \tilde{\chi}_{[ij]}$. One can argue [5] that in the large N limit its effective dynamics should be the same as the one of a gauge theory with the same $\mathrm{U}(N)$ gauge group but with matter in the adjoint representation. We will indeed reproduce this large N behavior. Here we wish on the other hand to determine the effective superpotential at finite N, to see whether a strict equivalence can be established, as for instance it was shown in [6] for the theory with fundamental matter.

We find that the theory with antisymmetric matter is actually more subtle, and has subleading (in N) corrections which yield a superpotential such that the N-dependence does *not* factorize. This behavior is reminiscent of SO / Sp theories with matter in the symmetric/antisymmetric representation [7]–[11] (i.e. not in the adjoint). Actually we will see that not only the $\mathrm{U}(N)$ theory with antisymmetric matter can be solved in a very similar way, but there is actually a precise map between this theory and an $\mathrm{Sp}(\tilde{N})$ theory with antisymmetric matter, with $\tilde{N} = 2N-2$ (we use the convention where $\tilde{N}$ is always even).

Note that in the theory we consider there is no field in the adjoint present. The theories where matter both in the adjoint and in the (anti)symmetric representations are present have been considered in [12]–[14]. We will discuss the relation of those theories to the present one in the concluding discussion.

While preparing this paper for submission, we have received [15] where the same theory is considered. The approach is similar and the conclusions are consistent, though the focus is slightly different.

2. Generalized Konishi anomaly

We begin by reviewing the chiral ring of the $\mathrm{U}(N)$ gauge theory with matter in the antisymmetric representation. The group action on the present representation:

$$\begin{aligned}(\mathcal{W}^a_\alpha T_a \chi)^{ij} &= \mathcal{W}_\alpha{}^i{}_k \chi^{kj} + \mathcal{W}_\alpha{}^j{}_k \chi^{ik}\,,\\ &= (\mathcal{W}_\alpha \chi)^{ij} + (\chi \mathcal{W}^T_\alpha)\,,\end{aligned} \tag{2.1}$$

leads to the following relation in the chiral ring:

$$\mathcal{W}_\alpha \chi = -\chi \mathcal{W}^T_\alpha\,, \tag{2.2}$$

and similarly for $\tilde{\chi}$, $\tilde{\chi}\mathcal{W}_\alpha = -\mathcal{W}^T_\alpha \tilde{\chi}$, so that $\mathcal{W}_\alpha$ commutes with the pair $\chi\tilde{\chi}$. Note that for future convenience we write $\mathcal{W}_\alpha$ in the fundamental representation, i.e. as $N \times N$ matrices.

As usual, non trivial operators in the chiral ring cannot have more than two $\mathcal{W}_\alpha$. On the other hand, arbitrary powers of χ and $\tilde{\chi}$ can be multiplied, provided that they are alternated, $\ldots \chi^{ij}\tilde{\chi}_{jk}\chi^{kl}\tilde{\chi}_{lp}\ldots$. Hence the most general gauge-variant operators in the chiral ring are:

$$\mathcal{W}^n(\chi\tilde{\chi})^k\,, \qquad \mathcal{W}^n(\chi\tilde{\chi})^k\chi\,, \qquad \tilde{\chi}\mathcal{W}^n(\chi\tilde{\chi})^k\,, \qquad \tilde{\chi}\mathcal{W}^n(\chi\tilde{\chi})^k\chi\,, \tag{2.3}$$

with $n \leq 2$ and k arbitrary. It is much like the theory with the adjoint, in the sense that one can construct independent polynomials in χ and $\tilde{\chi}$. Note that this is not the case in the theory with the fundamental, where the meson operator is the only independent composite operator in the chiral ring.

Gauge invariant operators in the chiral ring are just given by:

$$\mathrm{tr}\, \mathcal{W}^n(\chi\tilde{\chi})^k\,. \tag{2.4}$$

We now take a generic tree level superpotential:

$$W_{\mathrm{tree}} = \sum_{k=1}^{m} \frac{1}{k} g_k\, \mathrm{tr}(\chi\tilde{\chi})^k\,. \tag{2.5}$$

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Taking simply the equation for the anomalous U(1) rotations of the superfield χ, we obtain:

$$\bar{D}^2(\chi^\dagger e^V \chi) = \sum_{k=1}^{m} g_k \operatorname{tr}(\chi\tilde{\chi})^k - (N-2)S\,, \tag{2.6}$$

with as usual,

$$S = -\frac{1}{32\pi^2}\operatorname{tr}\mathcal{W}^\alpha\mathcal{W}_\alpha = \frac{1}{16\pi^2}\operatorname{tr}\lambda^\alpha\lambda_\alpha + \cdots\,. \tag{2.7}$$

Since in eq. (2.6), as soon as $m \geq 2$, several independent chiral operators appear, it is not possible to solve for $\langle\operatorname{tr}(\chi\tilde{\chi})^k\rangle$ in terms of $\langle S\rangle$ and g_k using simply this equation (as it was possible for the theory with the fundamental for instance). We need to derive more relations, in the spirit of [3] for the theory with the adjoint.

We will take more general variations, which read:

$$\delta\chi \equiv F = \mathcal{W}^n(\chi\tilde{\chi})^p\chi\,. \tag{2.8}$$

We will be only interested in the cases $n = 0, 2$, and will actually consistently impose that the gauge invariant operators with one $\mathcal{W}$ vanish. Note that for $n = 0 = p$ we recover the usual linear rotation.

For an arbitrary representation Φ^r, the general chiral ring relation following from the one-loop anomaly in the variation $\delta\Phi^r = F^r(\mathcal{W},\Phi)$ reads:

$$\frac{\partial W_{\text{tree}}(\Phi)}{\partial\Phi^r}F^r = -\frac{1}{32\pi^2}\mathcal{W}^{\alpha r}{}_s\mathcal{W}_\alpha{}^s{}_t\frac{\partial F^t}{\partial\Phi^r}\,. \tag{2.9}$$

Applying this general formula to the antisymmetric representation, we obtain:

$$\frac{\partial W_{\text{tree}}}{\partial\chi^{ij}}F^{ij} = -\frac{1}{32\pi^2}2\left(\mathcal{W}^{\alpha i}{}_j\mathcal{W}_\alpha{}^j{}_k\frac{\partial F^{kl}}{\partial\chi^{il}} + \mathcal{W}^{\alpha i}{}_j\mathcal{W}_\alpha{}^l{}_k\frac{\partial F^{jk}}{\partial\chi^{il}}\right). \tag{2.10}$$

Using (2.8), we obtain:

$$\begin{aligned}\frac{\partial F^{jk}}{\partial\chi^{il}} = \frac{1}{4}\sum_{r=0}^{p}\Big\{&[\mathcal{W}^n(\chi\tilde{\chi})^r]^j{}_i[(\chi\tilde{\chi})^{p-r}]^k{}_l - [\mathcal{W}^n(\chi\tilde{\chi})^r]^k{}_i[(\chi\tilde{\chi})^{p-r}]^j{}_l - \\ &- [\mathcal{W}^n(\chi\tilde{\chi})^r]^j{}_l[(\chi\tilde{\chi})^{p-r}]^k{}_i + [\mathcal{W}^n(\chi\tilde{\chi})^r]^k{}_l[(\chi\tilde{\chi})^{p-r}]^j{}_i\Big\}.\end{aligned} \tag{2.11}$$

For reference, we also write:

$$\frac{\partial F^{kl}}{\partial\chi^{il}} = \frac{1}{4}\sum_{r=0}^{p}\left\{\operatorname{tr}(\chi\tilde{\chi})^{p-r}[\mathcal{W}^n(\chi\tilde{\chi})^r]^k{}_i + \operatorname{tr}\mathcal{W}^n(\chi\tilde{\chi})^r[(\chi\tilde{\chi})^{p-r}]^k{}_i\right\} - \frac{1}{2}(p+1)[\mathcal{W}^n(\chi\tilde{\chi})^p]^k{}_i\,. \tag{2.12}$$

As a consistency check, we note that taking $n = p = 0$ we obtain $\frac{\partial\chi^{jk}}{\partial\chi^{il}} = \frac{1}{2}(\delta^j_i\delta^k_l - \delta^k_i\delta^j_l)$ and $\frac{\partial\chi^{kl}}{\partial\chi^{il}} = \frac{1}{2}(N-1)\delta^k_i$, and thus from (2.10) we recover (2.6).

With the general variation, we get:

$$\begin{aligned}\sum_{k=1}^{m} g_k\operatorname{tr}\mathcal{W}^n(\chi\tilde{\chi})^{k+p} = -\frac{1}{32\pi^2}\Bigg[&\frac{1}{2}\sum_{r=0}^{p}\{\operatorname{tr}(\chi\tilde{\chi})^{p-r}\operatorname{tr}\mathcal{W}^{n+2}(\chi\tilde{\chi})^r + \operatorname{tr}\mathcal{W}^n(\chi\tilde{\chi})^r\operatorname{tr}\mathcal{W}^2(\chi\tilde{\chi})^{p-r}\} - \\ &- 2(p+1)\operatorname{tr}\mathcal{W}^{n+2}(\chi\tilde{\chi})^p\Bigg].\end{aligned}$$

We thus have two sets of equations, for $n = 2$ and for $n = 0$:

$$\sum_{k=1}^{m} g_k \operatorname{tr} \mathcal{W}^2 (\chi\tilde{\chi})^{k+p} = -\frac{1}{32\pi^2}\frac{1}{2}\sum_{r=0}^{p} \operatorname{tr} \mathcal{W}^2 (\chi\tilde{\chi})^r \operatorname{tr} \mathcal{W}^2 (\chi\tilde{\chi})^{p-r} , \tag{2.13}$$

$$\sum_{k=1}^{m} g_k \operatorname{tr} (\chi\tilde{\chi})^{k+p} = -\frac{1}{32\pi^2}\left[\sum_{r=0}^{p} \operatorname{tr} \mathcal{W}^2 (\chi\tilde{\chi})^r \operatorname{tr} (\chi\tilde{\chi})^{p-r} - 2(p+1) \operatorname{tr} \mathcal{W}^2 (\chi\tilde{\chi})^p\right] \tag{2.14}$$

In order to obtain two closed equations for two generating functions of gauge invariant operators (the resolvents), we multiply both sides of the above equations by z^{-2p-2} and sum over all $p \geq 0$. We thus obtain from (2.13):

$$\sum_{k=1}^{m} g_k \operatorname{tr} \frac{\mathcal{W}^2 (\chi\tilde{\chi})^k}{z^2 - \chi\tilde{\chi}} = -\frac{1}{32\pi^2}\frac{1}{2}\left(z \operatorname{tr} \frac{\mathcal{W}^2}{z^2 - \chi\tilde{\chi}}\right)^2 . \tag{2.15}$$

We thus define the resolvent:

$$R(z) = -\frac{1}{32\pi^2} z \operatorname{tr} \frac{\mathcal{W}^2}{z^2 - \chi\tilde{\chi}} , \tag{2.16}$$

which has the usual behavior $R \sim \frac{S}{z}$ for large z. Note a technical subtlety: in the adjoint case, we could write (2.13) for $p = -1$ (or equivalently, for $\delta\Phi^i_j = \delta^i_j$), and that would correspond to a simple pole in z on the left hand side. Here this term is not present, and this is why we have to be slightly more subtle in the definition of $R(z)$.[1]

The equation for $R(z)$ reads:

$$\begin{aligned}
\frac{1}{2}R(z)^2 &= -\frac{1}{32\pi^2}\sum_{k=1}^{m} g_k \operatorname{tr} \frac{\mathcal{W}^2 (\chi\tilde{\chi})^k}{z^2 - \chi\tilde{\chi}} \\
&= -\frac{1}{32\pi^2}\sum_{k=1}^{m} g_k \operatorname{tr} \mathcal{W}^2 \frac{(\chi\tilde{\chi})^k - z^{2k} + z^{2k}}{z^2 - \chi\tilde{\chi}} \\
&= -\frac{1}{32\pi^2}\sum_{k=1}^{m} g_k z^{2k-1}\, z \operatorname{tr} \frac{\mathcal{W}^2}{z^2 - \chi\tilde{\chi}} - \frac{1}{32\pi^2}\sum_{k=1}^{m} g_k \operatorname{tr} \mathcal{W}^2 \frac{(\chi\tilde{\chi})^k - z^{2k}}{z^2 - \chi\tilde{\chi}} \\
&= R(z)W'(z) + \frac{1}{2}f(z) .
\end{aligned} \tag{2.17}$$

We thus see that, since we are assigning a power of z to every field, we need to define a superpotential function of degree $2m$, $W(z) = \sum_{k=1}^{m} \frac{1}{2k} g_k z^{2k}$. The polynomial $f(z)$ is in his turn of degree $2m - 2$.

The solution to the above equation is:

$$R(z) = W'(z) - \sqrt{W'(z)^2 + f(z)} \equiv W'(z) - y(z) . \tag{2.18}$$

Under the square root we have a polynomial of degree $4m - 2$, so that typically $y(z)$, and thus $R(z)$ will have $2m-1$ cuts on the complex plane. Note that until now, the structure is very similar to the $\mathrm{U}(N)$ theory with adjoint matter and an even tree level superpotential.

[1] In [15], this subtlety is treated in a different, but consistent, way.

We now sum the eqs. (2.14). We obtain:

$$\sum_{k=1}^{m} g_k \operatorname{tr} \frac{(\chi\tilde{\chi})^k}{z^2 - \chi\tilde{\chi}} = -\frac{1}{32\pi^2} z \operatorname{tr} \frac{1}{z^2 - \chi\tilde{\chi}} z \operatorname{tr} \frac{\mathcal{W}^2}{z^2 - \chi\tilde{\chi}} - \frac{1}{32\pi^2} z \frac{d}{dz} \operatorname{tr} \frac{\mathcal{W}^2}{z^2 - \chi\tilde{\chi}} . \tag{2.19}$$

The second term on the r.h.s. derives from the single trace term in the r.h.s. of (2.14).

Defining:

$$T(z) = z \operatorname{tr} \frac{1}{z^2 - \chi\tilde{\chi}} , \tag{2.20}$$

we obtain the following equation:

$$\sum_{k=1}^{m} g_k z^{2k-1} T(z) + \sum_{k=1}^{m} g_k \operatorname{tr} \frac{(\chi\tilde{\chi})^k - z^{2k}}{z^2 - \chi\tilde{\chi}} = T(z)R(z) + z \frac{d}{dz} \left(\frac{1}{z} R(z) \right) , \tag{2.21}$$

or:

$$W'(z)T(z) + c(z) = T(z)R(z) - \frac{1}{z} R(z) + R'(z) . \tag{2.22}$$

The leading order term of this equation, which is $1/z^2$, reproduces the relation (2.6). Note that unlike in the adjoint case, we have no $1/z$ term, which in that case reproduced the (traced) classical equations of motion. Here simply the classical equations cannot be traced.

We now realize that the equation for $T(z)$ is rather different from the one for matter in the adjoint, because of the two additional terms on the r.h.s. These two terms are reminiscent of those that appear for SO / Sp theories [8, 9] with matter in the adjoint (the $\frac{1}{z}R$ term) or in the symmetric/antisymmetric (the R' term). This analogy will be pushed further below.

For the time being, let us solve for $T(z)$, recalling that $y = W' - R$:

$$T(z) = \frac{\tilde{c}(z)}{y(z)} + \frac{1}{z} - \frac{d}{dz} \log y(z) , \tag{2.23}$$

where we have redefined the polynomial of degree $2m-2$ to be $\tilde{c}(z) = W''(z) - c(z) - \frac{1}{z} W'(z)$.

Before analyzing the general case, let us pause for a moment and consider the trivial case of $W_{\text{tree}} = m \operatorname{tr} \chi\tilde{\chi}$. Here the ordinary Konishi anomaly is sufficient to solve for the effective superpotential, but let us solve for the resolvents, in order to obtain the VEVs of all the gauge invariants in the SUSY vacuum.

Solving for $R(z)$, and fixing the constant f by requiring that for large z we have $R \sim \frac{S}{z}$, we obtain:

$$R(z) = mz - \sqrt{m^2 z^2 - 2mS} . \tag{2.24}$$

Also $T(z)$ is readily found, with c determined by the large z behavior $T \sim \frac{N}{z}$:

$$\begin{aligned} T(z) &= \frac{N}{\sqrt{z^2 - \frac{2S}{m}}} - \frac{z}{z^2 - \frac{2S}{m}} + \frac{1}{z} \\ &= \frac{N}{z} \left(1 + \frac{S}{mz^2} + \frac{3S^2}{2m^2 z^4} + \cdots \right) + \frac{1}{z} \left(-\frac{2S}{mz^2} - \frac{4S^2}{m^2 z^4} + \cdots \right) \end{aligned} \tag{2.25}$$

We thus find for the first two VEVs:

$$\operatorname{tr} \chi\tilde{\chi} = (N-2)\frac{S}{m}\,, \qquad \operatorname{tr}(\chi\tilde{\chi})^2 = \frac{3N-8}{2}\frac{S^2}{m^2}\,. \tag{2.26}$$

Contrary to what happens for the adjoint case, even in this simple setting the VEVs fail to display a common N-dependence.

Let us clarify here a possible source of confusion. If one applies the above formulas to the limiting cases of $N = 2$ or $N = 3$, where the antisymmetric is, respectively, the singlet and the (conjugate) fundamental, one quickly finds contradictions with the expected results of vanishing condensates for $N = 2$ and simple powers of the meson condensate for $N = 3$ (in the latter case the discrepancies arise at the next order). However, there is no contradiction, since one is actually computing the VEVs of chiral operators which can classically be expressed in terms of products of lower dimensional chiral operators. Now, these chiral ring relations can, and do, get non-perturbative quantum corrections (starting from the operator whose VEV is proportional to S^N, i.e. to a one-instanton contribution). The analogous phenomenon for the theory with adjoint matter is analyzed in [3] (see also [16]). Note that as long as such higher order operators do not appear in $W_{\rm tree}$, $W_{\rm eff}$ as computed in the following sections coincides strictly with the one computed, for instance for $N = 3$, by replacing the antisymmetric fields with fundamental ones.

3. Relation to an SO theory with adjoint matter

We are now going to see that, in order to correctly solve this theory, we have to embed it into an SO theory with adjoint matter, much in the spirit of the treatment by [10] for the Sp theory with antisymmetric matter.

Let us recall what the equations for the resolvent are in the SO theory with adjoint matter ϕ [8, 9]. We put a hat on SO quantities. Note that the tree level superpotential must be even because of antisymmetry of ϕ, $W_{\rm tree} = \sum_1^m \frac{1}{2k} h_{2k} \operatorname{tr} \phi^{2k}$. In terms of the resolvents:

$$\hat{R}(z) = -\frac{1}{32\pi^2} z \operatorname{tr} \frac{\mathcal{W}^2}{z^2-\phi^2}\,, \tag{3.1}$$

$$\hat{T}(z) = \operatorname{tr} \frac{z}{z^2-\phi^2}\,, \tag{3.2}$$

the two equations derived from the generalized Konishi anomalies read:

$$\frac{1}{2}\hat{R}(z)^2 = \hat{R}(z)\hat{W}'(z) + \frac{1}{2}\hat{f}(z)\,, \tag{3.3}$$

$$\hat{T}(z)\hat{R}(z) - \frac{2}{z}\hat{R}(z) = \hat{T}(z)\hat{W}'(z) + \hat{c}(z)\,, \tag{3.4}$$

with $\hat{f}(z)$ and $\hat{c}(z)$ two polynomials of degree $2m-2$ much similar to $f(z)$ and $c(z)$. We thus immediately see that (3.3) is the same as (2.17), while (3.4) features the R/z term, present also in (2.22).

Note that since traces of odd powers of adjoint SO matrices vanish, we could also write more familiarly:

$$\hat{T}(z) = \operatorname{tr} \frac{z}{z^2 - \phi^2} = \frac{1}{2}\left(\operatorname{tr}\frac{1}{z-\phi} + \operatorname{tr}\frac{1}{z+\phi}\right) = \operatorname{tr}\frac{1}{z-\phi}\,, \tag{3.5}$$

by using $\phi^T = -\phi$, and similarly for $\hat{R}(z)$. The $1/z^2$ terms of (3.4) lead to the ordinary Konishi anomaly relation:

$$\sum_{k=1}^{m} h_{2k} \operatorname{tr} \phi^{2k} = (N-2)\hat{S}\,. \tag{3.6}$$

Note that with this normalization the VY term will have a prefactor of $\frac{1}{2}(N-2)$. For later convenience we prefer to keep this normalization, although for purely SO considerations it would be preferable to rescale $\hat{S}$ to $2\hat{S}$.

Defining as before $\hat{y} = \hat{W}' - \hat{R}$, we can write the solution for $\hat{T}$ as:

$$\hat{T}(z) = \frac{\hat{\tilde{c}}(z)}{\hat{y}(z)} + \frac{2}{z}\,, \tag{3.7}$$

with $\hat{\tilde{c}}(z) = -\hat{c}(z) - \frac{1}{z}\hat{W}'(z)$. We thus observe that the structure of $\hat{T}(z)$ is the same as the one for a U(N) theory with adjoint matter, except for an additional pole at the origin. Moreover, this additional pole does not contribute at all to the expression for the VEVs $\langle \operatorname{tr} \phi^{2k}\rangle$. Indeed, it is possible to show that, for instance, in the one-cut case (classically unbroken SO(N)), all the above VEVs are proportional to the respective VEVs in the U(N) theory, with $N-2$ factorizing in front of them instead of N. The relation between SO and U theories with adjoint matter has been discussed in [17, 18, 19].

We are now going to embed the U(N) theory with the antisymmetric into an SO($\hat{N}$) theory with adjoint matter.

First of all, comparing (2.17) and (3.3), we can just equate $\hat{R}(z) = R(z)$, $\hat{W}'(z) = W'(z)$ and $\hat{f}(z) = f(z)$, and thus $\hat{y}(z) = y(z)$. Note that this also implies $\hat{S} = S$.

On the other hand, comparing, for instance, (2.23) and (3.7), we obtain a non-trivial relation:

$$\hat{T}(z) = 2T(z) + \frac{d}{dz}\log y(z)^2\,, \qquad\qquad \hat{\tilde{c}}(z) = 2\tilde{c}(z)\,. \tag{3.8}$$

Remembering that $y^2(z)$ is a polynomial of degree $4m-2$, we can immediately fix the relation between $\hat{N}$ and N:

$$\hat{N} = \frac{1}{2\pi i}\oint_{C_\infty} \hat{T}(z)dz = \frac{2}{2\pi i}\oint_{C_\infty} T(z)dz + \frac{1}{2\pi i}\oint_{C_\infty}\frac{d}{dz}\log y(z)^2 dz = 2N + 4m - 2\,, \tag{3.9}$$

so that the U(N) theory is mapped to a SO($2N+4m-2$) theory.

More importantly, we find that in the unbroken U(N) vacuum, we have:

$$\hat{N}_i = \frac{1}{2\pi i}\oint_{C_i} \hat{T}(z)dz = \frac{2}{2\pi i}\oint_{C_i} T(z)dz + \frac{1}{2\pi i}\oint_{C_i}\frac{d}{dz}\log y(z)^2 dz = 2\,, \tag{3.10}$$

since $\frac{d}{dz}\log y(z)^2$ has a pole at both edges of every cut (or a pole of residue 2 if the cut factorizes into a simple zero), and C_i circles around the ith cut not containing the origin.

We thus conclude that in order to study the unbroken vacuum of the $U(N)$ theory, we need to study the SO theory with symmetry breaking pattern:

$$\mathrm{SO}(2N+4m-2) \to \mathrm{SO}(2N+2) \times \mathrm{U}(2)^{m-1}\,. \tag{3.11}$$

More generally we expect a symmetry breaking pattern:

$$\mathrm{U}(N) \to \mathrm{U}(N_0) \times \mathrm{Sp}(N_1) \times \cdots \times \mathrm{Sp}(N_{m-1})\,, \tag{3.12}$$

with $N = N_0 + \sum_1^{m-1} N_i$, to map to:[2]

$$\mathrm{SO}(2N+4m-2) \to \mathrm{SO}(2N_0+2) \times \mathrm{U}(N_1+2) \times \cdots \times \mathrm{U}(N_{m-1}+2)\,. \tag{3.13}$$

Much in the same way as in Sp with antisymmetric matter [10], even in the simple case of the classically unbroken gauge group, we need to deal with a multicut solution. Note indeed that the unbroken gauge group scenario can be thought of as actually displaying $\mathrm{Sp}(0)$ factors, which have been studied in [20, 19]. This is a first hint that the N-dependence will not factorize in front of the effective superpotential, as indeed it is the case in the above mentioned case [7]–[9].

4. Effective superpotential for a quartic interaction

From now on, we will specialize to a theory with a quartic tree level superpotential:

$$W(z) = \frac{1}{2}mz^2 + \frac{1}{4}\lambda z^4\,. \tag{4.1}$$

The mapping is thus to a $\mathrm{SO}(2N+6)$ theory with symmetry breaking pattern $\mathrm{SO}(2N+6) \to \mathrm{SO}(2N+2) \times \mathrm{U}(2)$. The function $y(z)$ will thus have a cut around the origin and two symmetric cuts around the classical extrema of the superpotential, $\pm\sqrt{-\frac{m}{\lambda}}$.

We first want to compare the VEVs of the operators which appear in the tree level superpotential. They correspond to the $1/z^3$ and $1/z^5$ terms in the expansions of $T(z)$ and $\hat{T}(z)$. What we need to do is to find the corresponding terms in the expansion of $\frac{d}{dz}\log y(z)^2$. The leading behavior of f is $f \sim -2\lambda S z^2$ (it is fixed imposing $R \sim S/z$ for large z), thus we find:

$$\frac{d}{dz}\log y(z)^2 = \frac{6}{z} - \frac{4m}{\lambda z^3} + \left(\frac{4m^2}{\lambda^2} + \frac{8S}{\lambda}\right)\frac{1}{z^5} + \ldots\,, \tag{4.2}$$

so that we find:

$$\langle \mathrm{tr}\, \chi\tilde{\chi}\rangle = \frac{1}{2}\langle \mathrm{tr}\, \phi^2\rangle + \frac{2m}{\lambda}\,, \tag{4.3}$$

$$\langle \mathrm{tr}(\chi\tilde{\chi})^2\rangle = \frac{1}{2}\langle \mathrm{tr}\, \phi^4\rangle - \frac{2m^2}{\lambda^2} - \frac{4S}{\lambda}\,. \tag{4.4}$$

[2] One might be worried that the total number of vacua does not match. Indeed we have 2^m times more vacua in the second theory. A similar controversy is discussed and solved in [10]. Here we effectively hide it with our normalization of $\hat{S}$.

The corrections independent of S take care of the classical part of the VEVs (which is present in the SO theory but absent in the U(N) theory), while the correction linear in S is actually related to the matching of the scales of the two theories, as we discuss below.

What we are left to do is a quite laborious procedure. We should solve the SO($2N+6$) theory in terms of the two glueball superfields corresponding to the two low-energy gauge groups, extremize its effective superpotential, relate the latter to the U(N) effective superpotential through the relations above (and the relation between the holomorphic scales), and eventually integrate in the glueball superfield of the U(N) theory.

Instead, we will begin with a discussion on the scales of the theories involved, which clarifies when the subtleties related to the effective multi-cut solution set in (see [19] for a similar discussion in the Sp context).

First of all, let us see how (4.4) determines the matching of the scale Λ_h of the U(N) theory with the scale $\hat{\Lambda}_h$ of the SO($2N+6$). These are both high energy scales, related to the beta function of the theories with matter present. From (4.4) we can determine:

$$W_{\rm eff}^{U} = W_{\rm eff}^{SO} - 2S\log\lambda\mu + \frac{m^2}{\lambda}\,, \tag{4.5}$$

where μ is, say, the renormalization scale. Let us now consider that the scale dependent piece of $W_{\rm eff}$ reads, in general:

$$W_{\rm eff} = -\beta_0 S\log\frac{\Lambda_h}{\mu}\,. \tag{4.6}$$

This is basically the tree-level plus one-loop contribution to the superpotential, subtracted in order to write the effective superpotential in terms of S (i.e. after integrating it in). Thus the relation between the scales becomes:

$$-(2N+2)S\log\frac{\Lambda_h}{\mu} = -(2N+4)S\log\frac{\hat{\Lambda}_h}{\mu} - 2S\log\lambda\mu\,, \tag{4.7}$$

where we have used that $\beta_0^U = 3N-(N-2)$ and $\beta_0^{SO} = 2(2N+4)$, paying attention to the normalization we use on the SO side. We find the relation:

$$\Lambda_h^{2N+2} = \hat{\Lambda}_h^{2N+4}\lambda^2\,. \tag{4.8}$$

We now want to derive another relation, expressing the two low-energy scales $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ of the SO theory in terms of the unique low-energy scale Λ of the U(N) theory. In order to do this, we have to match the scales using the mass m of the matter field and, on the SO side, the VEV $|\phi|^2 = \frac{m}{\lambda}$ inducing gauge symmetry breaking (we will neglect all numerical factors in the following).

In the U(N) theory, the matching is straightforward:

$$\Lambda^{3N} = \Lambda_h^{2N+2}m^{N-2}\,. \tag{4.9}$$

In the SO($2N+6$) theory, we have to introduce the intermediate scales $\hat{\Lambda}_{0,int}$ and $\hat{\Lambda}_{1,int}$ of the SO($2N+2$) and U(2) theories with the adjoint field, respectively:

$$\hat{\Lambda}_{0,int}^{4N} = \hat{\Lambda}_h^{4N+8}\frac{\lambda^4}{m^4}\,, \qquad \hat{\Lambda}_{1,int}^{4} = \hat{\Lambda}_h^{4N+8}\frac{\lambda^{2N+2}}{m^{2N+2}}\,. \tag{4.10}$$

From those, we can get the relation to the scales at low energies:

$$\hat{\Lambda}_0^{6N} = \hat{\Lambda}_{0,int}^{4N} m^{2N} = \hat{\Lambda}_h^{4N+8} \lambda^4 m^{2N-4} = \Lambda_h^{4N+4} m^{2N-4} = \Lambda^{6N} , \qquad (4.11)$$

$$\hat{\Lambda}_1^6 = \hat{\Lambda}_{1,int}^4 m^2 = \hat{\Lambda}_h^{4N+8} \frac{\lambda^{2N+2}}{m^{2N}} = \Lambda_h^{4N+4} \frac{\lambda^{2N-2}}{m^{2N}} = \Lambda^{6N} \frac{\lambda^{2N-2}}{m^{4N-4}} . \qquad (4.12)$$

We thus find that $\hat{\Lambda}_0 = \Lambda$, while $\hat{\Lambda}_1^3 \propto \Lambda^{3N}$, i.e. in the effective superpotential the first contribution originally coming from the effective gauge dynamics in the U(2) factor looks like a one-instanton contribution in the U(N) effective theory. In other words, the effects related to the fact that $y(z)$ does not really factorize and has three cuts instead of one will appear in $W_{\rm eff}(S)$ only at order S^N and beyond. This is much like in the Sp with antisymmetric case, where the "discrepancies" between the field theory expectations and the naive matrix model computation set in at order S^h [7].

At this point, we could proceed to use the above information to get, for instance, the effective superpotential $W_{\rm eff}(\Lambda, m, \lambda)$ of the U(N) theory in the following way: first of all write the function y in terms of 3 parameters; relate those parameters to $\frac{m}{\lambda}$, S_0 and S_1 by using for the latter two their definitions in terms of contour integrals; write the expression for $\hat{T}(z)$, and determine the coefficients of $\hat{\tilde{c}}(z)$ by imposing the residues around the cuts corresponding to the size of the classical gauge groups; expand $\hat{T}(z)$ to find the expressions for $\langle \mathrm{tr}\, \phi^2 \rangle$ and $\langle \mathrm{tr}\, \phi^4 \rangle$; integrate to find an expression[3] for $W_{\rm eff}^{SO}$, to which one adds the relevant VY pieces; extremize this expression with respect to S_0 and S_1; substitute the expressions for $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ in terms of Λ, and eventually obtain $W_{\rm eff}^U$ using (4.5).

Another option, if we are satisfied in dealing case by case, is to analyze the factorization of the Seiberg-Witten curve, which yields the result for, say, $\langle \mathrm{tr}\, \phi^2 \rangle$ directly in terms of $\hat{\Lambda}_h$. This is the route effectively employed in [10]. At this effect, one could use the results of [21] (see also [22]).[4]

Here our main intention is to comment on the N-behavior of the effective superpotential. We can thus content ourselves in deriving the N-dependence for the first few terms in the S expansion, and consider that N is sufficiently large so that the subtleties associated to the non-factorization of y have no influence. In other words, as far as we are concerned we can just blindly consider the U(N) theory and suppose that y factorizes to a one-cut solution.[5] We are actually going to argue that the VEVs have the same expression as the VEVs of the Sp theory with antisymmetric matter and an even tree level superpotential (in the unbroken phase). Since it is known explicitly that in the latter case the N-dependence does not factorize, we thus conclude that also in the U(N) theory with antisymmetric matter the N-dependence does not factorize. Of course, in the large N-limit, the leading N behavior will reproduce the effective superpotential for the adjoint matter.

[3] Alternatively, we could obtain $W_{\rm eff}^{SO}$ by going through the free energy of the associated matrix model, obtained by expanding $R(z)$ rather than $\hat{T}(z)$.

[4] Note that a direct computation of the effective superpotential for U(N) with antisymmetric matter for low enough N could also be done along the lines of [23, 7], using the results on s-confining SU(N) theories with antisymmetric tensors and fundamentals found in [24].

[5] From the SO perspective, this amounts to freezing the dynamics of the U(2) factor by taking $S_1 = 0$, which implies $\hat{\Lambda}_1 = 0$.

Let us thus compute $W_{\mathrm{eff}}(S)$ at orders lower than S^N by supposing that the function $y(z)$ factorizes in the following way:

$$y(z) = \lambda(z^2 - a^2)\sqrt{z^2 - b^2}\,. \tag{4.13}$$

The relation between the constants a and b to the other data of the problem are obtained by remembering that $y^2 = W'^2 + f$, with $f = -2\lambda S z^2 + f_0$. We obtain:

$$\frac{m}{\lambda} = -a^2 - \frac{1}{2}b^2\,, \qquad 2\frac{S}{\lambda} = \frac{1}{4}b^2 - a^2 b^2\,. \tag{4.14}$$

Note that the second expression can alternatively be obtained by performing a contour integral of $R(z)$ around the cut. These relations can be inverted to:

$$a^2 = -\frac{m}{3\lambda}\left(2 + \sqrt{1 + 6\frac{\lambda S}{m^2}}\right), \tag{4.15}$$

$$b^2 = \frac{2m}{3\lambda}\left(-1 + \sqrt{1 + 6\frac{\lambda S}{m^2}}\right). \tag{4.16}$$

Correctly, for small $\frac{\lambda S}{m^2}$, we recover $a^2 \sim -\frac{m}{\lambda}$ and $b^2 \sim 2\frac{S}{m}$.

We can now write:

$$T(z) = \frac{c_2 z^2 + c_0}{(z^2 - a^2)\sqrt{z^2 - b^2}} + \frac{1}{z} - \frac{1}{z - a} - \frac{1}{z + a} - \frac{1}{2}\frac{1}{z - b} - \frac{1}{2}\frac{1}{z + b}\,. \tag{4.17}$$

The first coefficient c_2 is easily fixed by asking that:

$$\frac{1}{2\pi i}\oint_{C_\infty} T(z)dz = c_2 - 2 = N\,. \tag{4.18}$$

The second coefficient c_0 is then fixed requiring that the contour integral around the cut also gives N or, alternatively, that the contour integrals around the poles yield zero. This second route is the most straightforward:

$$\frac{1}{2\pi i}\oint_{C_a} T(z)dz = \frac{(N+2)a^2 + c_0}{2a\sqrt{a^2 - b^2}} - 1 = 0\,. \tag{4.19}$$

We eventually end up with the following generating function:

$$T(z) = \frac{N+2}{\sqrt{z^2 - b^2}} - \frac{2}{z} - \frac{b^2}{z(z^2 - b^2)} + \frac{2a}{z^2 - a^2}\left(\frac{\sqrt{a^2 - b^2}}{\sqrt{z^2 - b^2}} - \frac{a}{z}\right). \tag{4.20}$$

We immediately see that the first term has the same structure of the generating function for VEVs in the U(N) with adjoint case, the second term only corrects the leading term giving the trace of the identity, and most importantly the remaining terms will give corrections which are N-independent and will be different at every order in $1/z$. Note that in the large N limit, at leading order, we recover

$$T(z) \sim \frac{N}{\sqrt{z^2 - b^2}}\,, \tag{4.21}$$

which is exactly the generating function for $\mathrm{U}(N)$ with adjoint matter. In this sense for large N and in the planar limit the effective superpotentials for the two theories will coincide,[6] as argued in [5]. However we will see that the subleading corrections are more subtle than argued there, since no N-dependence can be factorized at finite N.

By expanding the expression (4.20), we recover the VEVs:

$$\langle \mathrm{tr}(\chi\tilde{\chi}) \rangle = \frac{N}{2}b^2 + 2a\sqrt{a^2-b^2} - 2a^2\,, \tag{4.22}$$

$$\langle \mathrm{tr}(\chi\tilde{\chi})^2 \rangle = \frac{3N-2}{8}b^4 + 2a^3\sqrt{a^2-b^2} - 2a^4 + ab^2\sqrt{a^2-b^2}\,. \tag{4.23}$$

Using the expressions (4.15) and (4.16), one can rewrite the above VEVs exactly in terms of S, m and λ, and in principle it is possible to obtain by integration an exact expression for $W_{\mathrm{eff}}(S, m, \lambda)$. We will refrain from doing so here, also because the exact expression obtained in this way should not be trusted to all orders but only up to S^{N-1}. Here we will only compute the first few terms in the expansion in S, in order to display their N-dependence.

For instance, since $b^2 \propto S$, we can expand (4.22) in b^2:

$$\langle \mathrm{tr}(\chi\tilde{\chi}) \rangle = \frac{N-2}{2}b^2 - \frac{1}{4}\frac{b^4}{a^2} - \frac{1}{8}\frac{b^6}{a^4} + \cdots\,. \tag{4.24}$$

Using then the expansions:

$$b^2 = \frac{2S}{m}\left[1 - \frac{3}{2}\frac{\lambda S}{m^2} + \frac{9}{2}\left(\frac{\lambda S}{m^2}\right)^2 + \cdots\right], \qquad a^2 = -\frac{m}{\lambda}\left[1 + \frac{\lambda S}{m^2} + \cdots\right], \tag{4.25}$$

we find:

$$\langle \mathrm{tr}(\chi\tilde{\chi}) \rangle = (N-2)\frac{S}{m} - \frac{1}{2}(3N-8)\frac{\lambda S^2}{m^3} + \frac{1}{2}(9N-28)\frac{\lambda^2 S^3}{m^5} + \cdots\,. \tag{4.26}$$

We thus obtain for the effective superpotential:

$$W_{\mathrm{eff}} = (N-2)S\log\frac{m}{\mu} + \frac{1}{4}(3N-8)\frac{\lambda S^2}{m^2} - \frac{1}{8}(9N-28)\frac{\lambda^2 S^3}{m^4} + \cdots\,. \tag{4.27}$$

The same expression, except the first term, can be obtained starting from $\langle \mathrm{tr}(\chi\tilde{\chi})^2 \rangle$.

A quick look at (4.27) shows that the N-dependence does not factorize. The $(N-2)$ behavior is restricted to the term responsible for the threshold matching.

Let us see what happens in two different limiting cases. First, consider the large N limit. We can define the VEVs:

$$v_2 = \frac{1}{N}\langle \mathrm{tr}(\chi\tilde{\chi}) \rangle \simeq \frac{1}{2}b^2\,, \tag{4.28}$$

$$v_4 = \frac{1}{N}\langle \mathrm{tr}(\chi\tilde{\chi})^2 \rangle \simeq \frac{3}{8}b^4 = \frac{3}{2}v_2^2\,. \tag{4.29}$$

[6] In the large N limit, we can consistently work in the present one-cut approximation, since the corrections at order S^N are pushed to infinity.

As in [6], we can insert these two expressions in the relation following from the ordinary Konishi anomaly:

$$mv_2 + \frac{3}{2}\lambda v_2^2 = S\,, \qquad \text{(large } N) \tag{4.30}$$

so that, referring to the equivalent VEVs in the U(N) theory with adjoint matter, we have $v_2 = u_2(\frac{S}{2}, m, \lambda)$, and thus:

$$W_{\text{eff}}^{\text{anti}}(S, m, \lambda) = 2W_{\text{eff}}^{\text{adj}}(\frac{S}{2}, m, \lambda)\,. \tag{4.31}$$

Hence, the effective superpotential for the U(N) theory with the antisymmetric is indeed essentially equivalent, in the leading large N limit, to the one for the theory with the adjoint and an even tree level superpotential, as argued in [5].

The other limiting case we can check is when $N = 3$. In this case, we know that for U(3) the antisymmetric is actually the conjugate fundamental, so that the theory boils down to a familiar case, U(3) with a single flavor. However, in this case we can trust our expansion (4.26) only to order S^2, and we expect discrepancies at order S^3 and higher.

For $N = 3$, (4.26) reads:

$$W_{\text{eff}}^{N=3} = S \log \frac{m}{\mu} + \frac{1}{4}\frac{\lambda S^2}{m^2} + \frac{1}{8}\frac{\lambda^2 S^3}{m^4} + \cdots\,. \tag{4.32}$$

In order to compare with the known, exact, effective superpotential for the theory with flavors, we first need to compare the VEVs. It turns out that if we define $\chi^{ij} = \frac{1}{\sqrt{2}}\epsilon^{ijk}\tilde{Q}_k$ and $\tilde{\chi}_{ij} = -\frac{1}{\sqrt{2}}\epsilon_{ijk}Q^k$, we have that $\operatorname{tr}\chi\tilde{\chi} = Q\tilde{Q} \equiv X$ and $\operatorname{tr}(\chi\tilde{\chi})^2 = \frac{1}{2}X^2$. Thus the relation following from the Konishi anomaly reads:

$$mX + \frac{\lambda}{2}X^2 = S\,. \tag{4.33}$$

Expanding now the expression found for instance in [25, 26] and replacing $\frac{\lambda}{2}$ for the quartic coupling, we get:

$$W_{\text{eff}}^{\text{fund}} = S \log \frac{m}{\mu} + \frac{1}{4}\frac{\lambda S^2}{m^2} - \frac{1}{8}\frac{\lambda^2 S^3}{m^4} + \cdots\,. \tag{4.34}$$

We thus see that the S^2 term is indeed reproduced by (4.32), but, as expected, at order S^3 (4.32) starts to disagree with the exact expression above.

5. Relation to Sp($\tilde{N}$) with antisymmetric matter

Let us comment on the relation between effective superpotentials of different theories. The approach used here was reminiscent of the one used for the Sp(N) gauge theory with matter in the (traceful) antisymmetric representation. We now point out that the two effective superpotentials actually coincide, upon replacing N here with $\frac{N}{2} + 1$ on the Sp(N) side. Note that on the Sp(N) side an even classical superpotential is not the most general one. Here for simplicity we only consider a quartic W_{tree}.

First of all, let us note that we embed our U(N) theory in a SO($2N + 6$) theory with adjoint matter, while the Sp($\tilde{N}$) theory is embedded in a U($\tilde{N} + 6$) theory. The superpotential in a generic phase of the SO($2N + 6$) theory can be directly extracted from

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the superpotential of the $U(\tilde{N}+6)$ theory [19], by taking in the latter the two cuts not containing the origin to be symmetric, and identifying the two glueball fields associated to them. After that, the expressions for the VEVs are the same (and thus the effective superpotential) since the additional term in $T(z)$, see eq. (3.7), only affects the leading term in the expansion. The only redefinition in $W_{\rm eff}$ is to map the two dual Coxeter numbers, implying $2N_0 - 2 = \tilde{N}_0$ for the first factors ($SO(2N_0)$ and $U(\tilde{N}_0)$ respectively) and $N_1 = \tilde{N}_1 = \tilde{N}_2$ for the other factors (which are all unitary). In particular, when we have the breaking pattern $SO(2N+6) \to SO(2N+2) \times U(2)$, on the other side we have $U(\tilde{N}+6) \to U(\tilde{N}+2) \times U(2) \times U(2)$, with the glueballs of the last two factors identified. The relation is thus $N = \frac{\tilde{N}}{2}+1$, that is the dual Coxeter number of $U(N)$ mapped to the one of $Sp(\tilde{N})$.

The logarithmic correction to the generating functions $T(z)$ is also the same in the two cases under considerations. We thus conclude that the effective superpotential is the same, up to the identification between the dual Coxeter numbers.

Note the following subtlety. It is quite straightforward to understand that all terms up to S^{h-1} share the same numerical factors, based on the naive procedure outlined in the previous section. It is on the other hand less obvious that the $\mathcal{O}(S^h)$ corrections give also the same contributions. Indeed, on the $Sp(\tilde{N})$ side we have contributions from two additional $U(2)$ gauge groups, while on the $U(N)$ side the dynamics of only one $U(2)$ contributes, basically the diagonal one. However we are confident that upon extremization, the final expressions in terms of the scale of the original theory with classically unbroken gauge group indeed coincide. For instance, a direct comparison using the $U(3)$ and $Sp(4)$ theories is possible. Quite straightforwardly one can reproduce the exact superpotential for $Sp(4)$ with a quartic interaction[7] along the lines of [23, 7], and it coincides with the one for $U(3)$ with one flavor, even before integrating out the meson superfield.

6. Discussion

We have shown in this paper that the effective superpotential for a $U(N)$ theory with antisymmetric matter has an N-dependence that does not factorize. Thus it does not share the same "universal" functional form as the effective superpotential for $U(N)$ with adjoint matter, a functional form that was also shown to be reproduced by a $U(N)$ theory with fundamental matter [6]. On the other hand, we have shown that, for every N, the superpotential is the same as the one arising in a theory with $Sp(2N-2)$ gauge group and antisymmetric matter. Since the N-dependence does not factorize, these superpotentials are different functions of the glueball and couplings for every N.

The above results concern the vacuum with classically unbroken gauge group. It is amusing to note the following thing. In vacua where on the other hand the gauge group is classically maximally broken, $U(N) \to U(N_0) \times Sp(N_1) \times \ldots \times Sp(N_{m-1})$, the structure is expected to be much more regular. In particular, the structure of the effective superpotential (in terms of the glueballs S_i) will be a sum of terms, each of which displaying

[7] The $Sp(4)$ theory with antisymmetric matter has been solved in [7, 10], though only in the (actually more complicated) case where the classical superpotential is cubic.

a factorized dependence on N_i.[8] This is because in this case the (in)famous Sp(0) factors are absent. Of course, all factorized dependence on N_i is lost when the glueballs are integrated out. See [19] for the related Sp case.

We cannot refrain from speculating that the equivalences among effective superpotentials of theories differing by the gauge group and/or the matter content might be due to underlying duality relations. Also, it is intriguing that in all the cases considered, it is possible by a chain of mappings to relate the theory under consideration to a U(N) theory with adjoint matter (with some restrictions on the classical superpotential and symmetry breaking pattern). It is possible that the generic solution to this latter theory contains all the information to solve for all other theories, for classical groups and up to two index representations at least.

We have mentioned in the introduction that U(N) theories with both adjoint and antisymmetric matter have been analyzed [12]–[14]. These theories have the special feature that they are solved in terms of a cubic curve $y^3(z) = \ldots$. While in our case the solution had a more traditional quadratic curve associated to it, it should in principle be possible to relate at least some of the results in both theories. Indeed, consider the theory with the adjoint and a simple tree level superpotential like:

$$W_{\rm tree} = \frac{1}{2} M \,\mathrm{tr}\, \Phi^2 + m \,\mathrm{tr}\, \chi\tilde{\chi} + g \,\mathrm{tr}\, \Phi\chi\tilde{\chi}\,. \tag{6.1}$$

This is the simplest case of [12]–[14], where a generic superpotential for Φ is considered. However it should be exactly equivalent to our case with a quartic superpotential, by classically integrating out Φ (by holomorphy, we can always take M to be very large in this step) and identifying $\lambda \equiv -\frac{g^2}{M}$. It would be interesting to check that the superpotentials indeed coincide, as it is the case for the theories with fundamental matter [25].

Finally, let us comment on a seemingly straightforward generalization, that is to U(N) with symmetric matter. Here we expect the theory to be mapped to a SO theory with symmetric matter. However, this theory in the unbroken phase cannot be solved in exactly the same way as the companion Sp theory. Indeed, the trivial factors which arise are SO(0) factors, and those cannot be dealt with in the same way as the Sp(0) ones. See [19] for a discussion on how to deal with this, based on geometric transitions. It would be nice to have an understanding of this case in pure gauge theoretic terms.

Acknowledgments

I would like to thank A. Armoni and G. Ferretti for interesting discussions and correspondence. This work is supported in part by the "Actions de Recherche Concertées" of the "Direction de la Recherche Scientifique - Communauté Française de Belgique", by a "Pôle d'Attraction Interuniversitaire" (Belgium), by IISN-Belgium (convention 4.4505.86) and by the European Commission RTN programme HPRN-CT-00131. The author is a Postdoctoral Researcher of the Fonds National de la Recherche Scientifique (Belgium).

[8] More precisely, N_0 factorizes for the U(N_0) factors and $\frac{N_i}{2}+1$ factorizes for the Sp factors.

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